

Chapter Series Solutions of differential Equations

Def .

A function $f(x)$ is said to be analytic at the point $x = x_0$ if and only if it has a Taylor series at $x = x_0$ which represents the function in some neighborhood of $x = x_0$.

Def. Point

If the coefficients functions $P(x)$ and $Q(x)$ in the equation

$y'' + P(x)y' + Q(x)y = 0$ are both **analytic** at $x = x_0$, then x_0 is said to be

an **ordinary point** of the equation. If at least one of the functions $P(x)$

and $Q(x)$ is not analytic at $x = x_0$ but if both $(x - x_0)P(x)$ and

$(x - x_0)^2 Q(x)$ are **analytic** at $x = x_0$, then x_0 is said to be a **regular**

singular point of the equation. If at least one of the functions $(x - x_0)P(x)$

and $(x - x_0)^2 Q(x)$ is not analytic at $x = x_0$, then x_0 is said to be a

irregular singular point of the equation.

Ex. Solve $y' + p(x)y = r(x)$ with $y(x_0) = y_0$

Ans. Express $y(x) = y(x_0) + y'(x_0)(x - x_0) + \frac{1}{2}y''(x_0)(x - x_0)^2 + ..$

The equation $y'(x_0) + p(x_0)y(x_0) = r(x_0)$ yields

$$y'(x_0) = -p(x_0)y(x_0) + r(x_0)$$

and $y''(x_0) = r'(x_0) - p'(x_0)y(x_0) - p(x_0)y'(x_0)$, etc.

Ex. Solve $y' + (1 + x^2)y = \sin x$ with $y(0) = a$

Ans. Express $y(x) = a + y'(0)x + \frac{1}{2}y''(0)x^2 + \dots$

The equation $y' + (1+x^2)y = \sin x$ yields

$$y' = -(1+x^2)y + \sin x, \quad y'' = -(1+x^2)y' - 2xy + \cos x,$$

$$y''' = -2xy' - (1+x^2)y'' - 2y - 2xy' - \sin x, \text{ etc.}$$

Taking $x=0$ into the above equations yields

$$y'(0) = -y(0) = -a, \quad y''(0) = -y'(0) + 1 = 1 + a,$$

$$y'''(0) = -(1+a) - 2a = -(1+3a), \text{ etc.}$$

Finally, $y(x) = a - ax + (1+a)\frac{x^2}{2!} - (1+3a)\frac{x^3}{3!} + \dots$

Ex. Solve $y' + 4xy = 3e^{x-1}$ with $y(1) = 1$

Ans. Express $y(x) = 1 + y'(x_0)(x-1) + \frac{1}{2}y''(1)(x-1)^2 + \dots$

The equation $y' + 4xy = 3e^{x-1}$ yields

$$y' = -4xy + 3e^{x-1}, \quad y'' = -4y - 4xy' + 3e^{x-1}, \quad y''' = -8y' - 4xy'' + 3e^{x-1}, \text{ etc.}$$

Taking $x=1$ into the above equations yields

$$y'(1) = -4y(1) + 3 = -1, \quad y''(1) = -4 + 4 + 3 = 3, \quad y'''(1) = 8 - 12 + 3 = -1, \text{ etc.}$$

Finally, $y(x) = 1 - (x-1) + \frac{3}{2}(x-1)^2 + \dots$

Ex. Solve $y'' + xy' + (1-x^2)y = x$ with $y(0) = a$ and $y'(0) = b$

Ans. Express $y(x) = a + bx + \frac{1}{2}y''(0)x^2 + \dots$

The equation $y'' + xy' + (1-x^2)y = x$ yields

$$y'' = -xy' - (1-x^2)y + x, \quad y''' = -xy'' - (2-x^2)y' + 2xy + 1,$$

$$y^{iv} = -y'' - xy''' + 2xy' - (2-x^2)y'' + 2y + 2xy', \text{ etc.}$$

Taking $x=0$ into the above equations yields

$$y''(0) = -y(0) = -a, \quad y'''(0) = -2b + 1, \quad y^{iv}(0) = 3a + 2a = 5a$$

$$\text{Finally, } y(x) = a + bx - a \frac{x^2}{2!} + (1-2b) \frac{x^3}{3!} + 5a \frac{x^4}{4!} + \dots$$

HW. Find the first four terms in the power series solution of the following equations

$$1. (1+3x^2)y'' + 2xy' + 2xy = 1 \quad \text{with } y(0) = a, \quad y'(0) = b.$$

$$2. y'' + (x-x^2)y' + y = 0 \quad \text{with } y(0) = 2, \quad y'(0) = -3$$

$$3. (1-x)y'' + 2xy' - 2xy = 0 \quad \text{with } y(2) = 1, \quad y'(2) = 5$$

§ Homogeneous Solutions

$$\text{Ex. } y'' + \frac{2}{x}y' + \frac{3}{x(x-1)^3}y = 0$$

The solution $y(x)$ is expressed as

$$y = |x-1|^r [a_0 + a_1(x-1) + a_2(x-1)^2 + \dots]$$

$$\text{Or } y = a_0(x-1)^r + a_1(x-1)^{1+r} + a_2(x-1)^{2+r} + \dots$$

$$\text{Ex. } y'' + xy' + y = 0$$

Ans. The point $x=0$ is an ordinary point of the equation.

So, the solution $y(x)$ is expressed as $y = \sum_{i=0}^{\infty} a_i x^i$.

$$\sum_{i=0}^{\infty} i(i-1)a_i x^{i-2} + \sum_{i=0}^{\infty} i a_i x^i + \sum_{i=0}^{\infty} a_i x^i = 0$$

$$\sum_{i=2}^{\infty} i(i-1)a_i x^{i-2} + \sum_{k=0}^{\infty} a_k (k+1)x^k = 0,$$

Let $i = j+2$, then $i=2 \rightarrow j=0$

$$\sum_{j=0}^{\infty} (j+2)(j+1)a_{j+2}x^j + \sum_{k=0}^{\infty} a_k (k+1)x^k = 0 \quad \text{or}$$

$$\sum_{j=0}^{\infty} [(j+2)(j+1)a_{j+2} + (j+1)a_j]x^j = 0, \text{ so } a_{j+2} = -\frac{j+1}{(j+2)(j+1)}a_j$$

Ex. $xy'' + y' + y = 0$

Ans. The equation can be expressed as $y'' + \frac{1}{x}y' + \frac{1}{x}y = 0$ which

indicates $x=0$ is a regular singular point and $x=1$ is an ordinary point.

So, the solution is expressed as $y = \sum_{i=0}^{\infty} a_i x^i$

$$x \sum_{i=0}^{\infty} i(i-1)a_i x^{i-2} + \sum_{i=0}^{\infty} i a_i x^{i-1} + \sum_{i=0}^{\infty} a_i x^i = 0 \text{ or}$$

$$x \sum_{i=2}^{\infty} i(i-1)a_i x^{i-2} + \sum_{i=1}^{\infty} i a_i x^{i-1} + \sum_{i=0}^{\infty} a_i x^i = 0$$

$$\sum_{i=2}^{\infty} i(i-1)a_i x^{i-1} + \sum_{i=2}^{\infty} i a_i x^{i-1} + a_1 + \sum_{i=0}^{\infty} a_i x^i = 0$$

$$\sum_{i=2}^{\infty} i^2 a_i x^{i-1} + a_1 + a_0 + \sum_{i=1}^{\infty} a_i x^i = 0, \quad \sum_{i=1}^{\infty} [(i+1)^2 a_{i+1} + a_i] x^i + a_1 + a_0 = 0$$

implies $a_1 = -a_0$ and $a_{i+1} = -\frac{1}{(i+1)^2} a_i, \quad i=1, \dots$

Or

$$a_1 = -a_0, \quad a_2 = \frac{1}{2^2} a_0, \quad a_3 = -\frac{1}{3^2} a_2 = -\frac{1}{(3!)^2} a_0, \quad a_4 = \frac{1}{(4!)^2} a_0$$

$$y(x) = a_0 \left[1 - x + \frac{x^2}{(2!)^2} - \frac{x^3}{(3!)^2} + \frac{x^4}{(4!)^2} + \dots \right]$$

Ex. Solve $y'' + 2xy' + (1+x^2)y = 0$ with $y(0) = 3$ and $y'(0) = -1$

Ans. $y(x) = \sum_{i=0}^{\infty} a_i x^i \rightarrow \sum_{i=2}^{\infty} i(i-1)a_i x^{i-2} + \sum_{i=1}^{\infty} 2i a_i x^i + \sum_{i=0}^{\infty} a_i x^i + \sum_{i=0}^{\infty} a_i x^{i+2} = 0$

$$\rightarrow \sum_{i=0}^{\infty} [(i+2)(i+1)a_{i+2} + a_i] x^i + \sum_{i=1}^{\infty} 2i a_i x^i + \sum_{i=0}^{\infty} a_i x^{i+2} = 0 \rightarrow$$

$$(2a_2 + a_0) + (6a_3 + a_1)x + \sum_{i=2}^{\infty} [(i+2)(i+1)a_{i+2} + a_i] x^i + 2a_1 x + \sum_{i=2}^{\infty} 2i a_i x^i + \sum_{i=0}^{\infty} a_i x^{i+2} = 0$$

$$\rightarrow (2a_2 + a_0) + (6a_3 + 3a_1)x + \sum_{i=2}^{\infty} [(i+2)(i+1)a_{i+2} + (1+2i)a_i] x^i + \sum_{i=0}^{\infty} a_i x^{i+2} = 0$$

Let $i = j+2$, then

$$(2a_2 + a_0) + (6a_3 + 3a_1)x + \sum_{j=0}^{\infty} [(j+4)(j+3)a_{j+4} + (5+2j)a_{j+2}]x^{j+2} + \sum_{i=0}^{\infty} a_i x^{i+2} = 0$$

$$(2a_2 + a_0) + (6a_3 + 3a_1)x + \sum_{j=0}^{\infty} [(j+4)(j+3)a_{j+4} + (5+2j)a_{j+2} + a_j]x^{j+2} = 0$$

$$\rightarrow a_2 = -\frac{1}{2}a_0, \quad a_3 = -\frac{1}{2}a_1, \quad a_{j+4} = -\frac{5+2j}{(j+4)(j+3)}a_{j+2} - \frac{1}{(j+4)(j+3)}a_j$$

$$\rightarrow a_4 = -\frac{5}{12}a_2 - \frac{1}{12}a_0 = \frac{1}{8}a_0, \quad a_5 = -\frac{7}{20}a_3 - \frac{1}{20}a_1 = \frac{1}{8}a_1,$$

$$a_6 = -\frac{9}{30}a_4 - \frac{1}{30}a_2 = -\frac{1}{48}a_0, \quad a_7 = -\frac{11}{42}a_5 - \frac{1}{42}a_3 = -\frac{1}{48}a_1$$

$$\text{So, } y(x) = a_0(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 - \frac{1}{48}x^6 + \dots) + a_1(x - \frac{1}{2}x^3 + \frac{1}{8}x^5 - \frac{1}{48}x^6 + \dots)$$

$$y'(x) = a_0(-x + \frac{1}{2}x^3 - \frac{1}{8}x^5 + \dots) + a_1(1 - \frac{3}{2}x^2 + \frac{5}{8}x^4 - \frac{6}{48}x^5 + \dots)$$

$$y(0) = 3 = a_0, \quad y'(0) = -1 = a_1$$

$$\text{Finally, } y(x) = 3(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 - \frac{1}{48}x^6 + \dots) - (x - \frac{1}{2}x^3 + \frac{1}{8}x^5 - \frac{1}{48}x^6 + \dots)$$

Ex. Solve $y'' + xy' + y = 0$ with $y(0) = 1$ and $y'(0) = 0$

$$\text{Sol. } y(x) = \sum_{i=0}^{\infty} a_i x^i \rightarrow \sum_{i=2}^{\infty} i(i-1)a_i x^{i-2} + \sum_{i=1}^{\infty} i a_i x^i + \sum_{i=0}^{\infty} a_i x^i = 0$$

$$\text{Let } i = j+2 \rightarrow \sum_{j=0}^{\infty} (j+2)(j+1)a_{j+2} x^j + \sum_{i=1}^{\infty} i a_i x^i + a_0 + \sum_{i=1}^{\infty} a_i x^i = 0$$

$$\rightarrow a_0 + 2a_2 + \sum_{j=1}^{\infty} [(j+2)(j+1)a_{j+2} + (j+1)a_j]x^j = 0$$

$$\rightarrow a_2 = -\frac{1}{2}a_0, \quad a_{j+2} = -\frac{1}{j+2}a_j, \quad \rightarrow a_2 = -\frac{1}{2}a_0, \quad a_3 = -\frac{1}{3}a_1,$$

$$a_4 = -\frac{1}{4}a_2 = \frac{1}{8}a_0, \quad a_5 = -\frac{1}{5}a_3 = \frac{1}{15}a_1, \quad a_6 = -\frac{1}{6}a_4 = -\frac{1}{48}a_0, \dots$$

$$y(x) = a_0(1 - \frac{x^2}{2} + \frac{x^4}{8} + \dots) + a_1(x - \frac{x^3}{3} + \frac{x^5}{15} + \dots)$$

$$y'(x) = a_0(-x + \frac{x^3}{2} + \dots) + a_1(1 - x^2 + \frac{x^4}{3} + \dots)$$

$$y(0) = 1 = a_0, \quad y'(0) = 0 = a_1 \rightarrow y(x) = 1 - \frac{x^2}{2} + \frac{x^4}{8} + \dots$$

Legendre differential equation

$$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0$$

$x = 0$ is an ordinary point, $y(x) = \sum_{i=0}^{\infty} a_i x^i$

$$\sum_{i=2}^{\infty} i(i-1)a_i x^{i-2} - \sum_{i=2}^{\infty} i(i-1)a_i x^i - 2 \sum_{i=1}^{\infty} i a_i x^i + \alpha(\alpha+1) \sum_{i=0}^{\infty} a_i x^i = 0$$

$$\sum_{j=0}^{\infty} (j+2)(j+1)a_{j+2} x^j - \sum_{i=2}^{\infty} i(i-1)a_i x^i - 2 \sum_{i=1}^{\infty} i a_i x^i + \alpha(\alpha+1) \sum_{i=0}^{\infty} a_i x^i = 0$$

$$\sum_{j=0}^{\infty} [(j+2)(j+1)a_{j+2} + \alpha(\alpha+1)a_j] x^j - \sum_{i=2}^{\infty} i(i+1)a_i x^i - 2a_1 x = 0$$

$$\rightarrow [2a_2 + \alpha(\alpha+1)a_0] + [6a_3 + (\alpha+2)(\alpha-1)a_1]x +$$

$$\sum_{j=2}^{\infty} \{(j+2)(j+1)a_{j+2} + \alpha(\alpha+1)a_j - j(j+1)a_j\} x^j = 0$$

$$\rightarrow 2a_2 + \alpha(\alpha+1)a_0 = 0, \quad 6a_3 + (\alpha+2)(\alpha-1)a_1 = 0$$

$$(j+2)(j+1)a_{j+2} + [\alpha(\alpha+1) - j(j+1)]a_j = 0$$

$$\rightarrow a_2 = -\frac{\alpha(\alpha+1)}{2}a_0, \quad a_3 = -\frac{(\alpha+2)(\alpha-1)}{6}a_1, \quad a_{j+2} = -\frac{(\alpha-j)(\alpha+1+j)}{(j+2)(j+1)}a_j$$

$$y(x) = (a_0 + a_2 x^2 + a_4 x^4 + \dots) + (a_1 x + a_3 x^3 + \dots)$$

Special cases:

1. $\alpha = 0$

$$\sum_{i=2}^{\infty} i(i-1)a_i x^{i-2} - \sum_{i=2}^{\infty} i(i-1)a_i x^i - 2 \sum_{i=1}^{\infty} i a_i x^i = 0$$

$$\sum_{j=0}^{\infty} (j+2)(j+1)a_{j+2} x^j - \sum_{i=2}^{\infty} i(i-1)a_i x^i - 2 \sum_{i=1}^{\infty} i a_i x^i = 0$$

$$\sum_{j=2}^{\infty} (j+2)(j+1)a_{j+2} x^j + 2a_2 + 6a_3 x - \sum_{i=2}^{\infty} i(i+1)a_i x^i - 2a_1 x = 0$$

$$\sum_{j=2}^{\infty} (j+1)[(j+2)a_{j+2} - j a_j] x^j + 2a_2 + (6a_3 - 2a_1)x = 0$$

$$a_2 = a_4 = a_6 = \dots = 0, \quad a_3 = \frac{1}{3}a_1, \quad a_{j+2} = \frac{j}{j+2}a_j,$$

$$a_5 = \frac{3}{5}a_3 = \frac{1}{5}a_1, \quad a_7 = \frac{5}{7}a_5 = \frac{1}{7}a_1$$

$$y(x) = a_0 + a_1 \left(x + \frac{x^3}{3} + \frac{x^5}{5} + \dots \right)$$

2. $\alpha = 1$

$$\sum_{i=2}^{\infty} i(i-1)a_i x^{i-2} - \sum_{i=2}^{\infty} i(i-1)a_i x^i - 2 \sum_{i=1}^{\infty} i a_i x^i + 2 \sum_{i=0}^{\infty} a_i x^i = 0$$

$$\sum_{j=0}^{\infty} (j+2)(j+1)a_{j+2} x^j - \sum_{i=2}^{\infty} i(i-1)a_i x^i - 2 \sum_{i=1}^{\infty} i a_i x^i + 2 \sum_{i=0}^{\infty} a_i x^i = 0$$

$$\sum_{j=0}^{\infty} [(j+2)(j+1)a_{j+2} + 2a_j] x^j - \sum_{i=2}^{\infty} i(i+1)a_i x^i - 2a_1 x = 0$$

$$\rightarrow [2a_2 + 2a_0] + 6a_3 x +$$

$$\sum_{j=2}^{\infty} \{(j+2)(j+1)a_{j+2} + 2a_j - j(j+1)a_j\} x^j = 0$$

$$\rightarrow 2a_2 + 2a_0 = 0, \quad 6a_3 = 0, \quad (j+1)a_{j+2} - (j-1)a_j = 0$$

$$\rightarrow a_2 = -a_0, \quad a_3 = 0, \quad a_{j+2} = \frac{j-1}{j+1} a_j, \quad a_4 = \frac{1}{3} a_2 = -\frac{1}{3} a_0, \quad a_6 = \frac{3}{5} a_4 = -\frac{1}{5} a_0$$

$$y(x) = a_0 \left(1 - x^2 - \frac{x^4}{3} - \frac{x^6}{5} - \dots\right) + a_1 x$$

Ex. Chebyshev equation

$$(1-x^2)y'' - xy' + m^2 y = 0$$

Ex. $y'' + \frac{2}{x}y' + \frac{3}{x(x-1)^3}y = 0$ has two singular points $x=0$ and $x=1$.

Except the two points, the equation is analytic everywhere.

1. Both 2 and $\frac{3x}{(x-1)^3}$ are analytic at $x=0$, so $x=0$ is a regular singular point.

2. $\frac{3}{(x-1)}$ is not analytic at $\frac{3x}{(x-1)^3}$, so $x=1$ is a irregular point.

Theorem

At an **ordinary point** $x = x_0$ of the differential equation

$$y'' + P(x)y' + Q(x)y = 0$$

Every solution is analytic, i.e., can be represented by a series of the form

$$y = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \dots$$

Moreover, the radius of convergence of each series solution is not less

than the distance from x_0 to the nearest singular point of the equation.

Theorem

At a **regular singular point** $x = x_0$ of the differential equation

$$y'' + P(x)y' + Q(x)y = 0$$

There is at least one solution which possesses an expansion of the form

$$y = |x - x_0|^r [a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots]$$

and this series will converge for $0 < |x - x_0| < R$, where R is not less than

the distance from x_0 to the nearest of the other singular points of the equation.

Theorem

At an **irregular singular point** $x = x_0$ of the differential equation

$$y'' + P(x)y' + Q(x)y = 0$$

there is in general **no solutions** with expansions consisting solely of powers of

$$x - x_0.$$

Ex $x^2 y'' + xy' + (x^2 - n^2)y = 0 \rightarrow y'' + \frac{1}{x}y' + (1 - \frac{n^2}{x^2})y = 0.$

$x = 0$ is a regular singular point.

Ex. $(1 - x^2)y'' - 2xy' + n(n+1)y = 0 \rightarrow y'' - \frac{2x}{1-x^2}y' + \frac{n(n+1)}{1-x^2}y = 0$

Both $x = 1$ and $x = -1$ are singular points.

At $x = 1$ both $(x-1)\frac{2x}{1-x^2}$ and $(x-1)^2\frac{n(n+1)}{1-x^2}$ are analytic

At $x = -1$ both $(x+1)\frac{2x}{1-x^2}$ and $(x+1)^2\frac{n(n+1)}{1-x^2}$ are analytic

So, both $x = 1$ and $x = -1$ are regular singular points.

Ex. $(1-x)y'' + 2(x-1)y' + xy = 0 \rightarrow y'' - 2y' + \frac{x}{1-x}y = 0$

$x=1$ is a regular singular point

Ex. $(x-1)^3 y'' + 3(x-1)^2 y' + y = 0 \rightarrow y'' + 3\frac{1}{x-1}y' + \frac{1}{(x-1)^3}y = 0$

$x=1$ is an irregular singular point

Ex. Consider the problem $x^2 y'' + 3xy' + 2y = 0$

$x=0$ is a regular singular point.

Let $z = \ln x$, then $x \frac{d}{dx} = \frac{d}{dz}$ and $x^2 \frac{d^2}{dx^2} = \frac{d}{dz}(\frac{d}{dz} - 1)$

$$x^2 y'' + 3xy' + 2y = 0 \rightarrow \frac{d}{dz}(\frac{d}{dz} - 1)y + 3\frac{dy}{dz} + 2y = 0 \rightarrow \frac{d^2 y}{dz^2} + 2\frac{dy}{dz} + 2y = 0$$

$$\rightarrow y(z) = c_1 e^{(-1+i)z} + c_2 e^{(-1-i)z} = e^{-z} [C_1 \cos z + C_2 \sin z]$$

$$y(x) = x^{-1} [C_1 \cos |\ln x| + C_2 \sin |\ln x|]$$

So, $y(x)$ cannot be expressed as $y(x) = \sum_{i=0}^{\infty} a_i x^i$ for $x=0$ being a regular singular point.

✂ The problem $(x-1)^2 y'' + 3(x+2)y' + 2y = 0$ can be shifted as

$$z^2 y'' + 3(z+3)y' + 2y = 0 \text{ by introducing } z = x-1.$$

HW. Identify the nature of the singular points of the following equations

1. $(1-x)^2 y'' + 2(x-1)y' + y = 0$,

2. $(x^2 - 4)y'' + (x+3)y' - 5(x+1)y = 0$

§ The Frobenius method

Let x_0 be a regular singular point of the equation

$$a(x)y'' + b(x)y' + c(x)y = 0$$

Then, in some interval $0 < |x - x_0| < d$, the solution will always possess at

least one solution of the form

$$y(x) = (x - x_0)^c \sum_{i=0}^{\infty} a_i (x - x_0)^i$$

where $a_0 \neq 0$ and c is a real or complex number. A second linearly independent solution of similar form will exist that may contain a logarithmic term, though with a different value of c and some other coefficients $b_0, b_1, \dots$ in place of the coefficients $a_0, a_1, \dots$. Taken together, these two solutions form a basis of solutions for the differential equation.

* Suppose the problem $y'' + \frac{1}{x} p(x) y' + \frac{1}{x^2} q(x) y = 0$ to have a regular

singular point $x = 0$. Then $y(x)$, $p(x)$ and $q(x)$ will be expressed as

$$y(x) = x^c \sum_{i=0}^{\infty} a_i x^i, \quad p(x) = p(0) + \sum_{i=1}^{\infty} \frac{1}{i!} \frac{d^i p}{dx^i}(0) x^i \equiv \sum_{i=0}^{\infty} p_i x^i,$$

$$q(x) = q(0) + \sum_{i=1}^{\infty} \frac{1}{i!} \frac{d^i q}{dx^i}(0) x^i \equiv \sum_{i=0}^{\infty} q_i x^i.$$

$$\sum_{i=0}^{\infty} (i+c)(i+c-1) a_i x^{i+c-2} + \left(\sum_{i=0}^{\infty} p_i x^i \right) \sum_{i=0}^{\infty} (i+c) a_i x^{i+c-2} + \left(\sum_{i=0}^{\infty} q_i x^i \right) \sum_{i=0}^{\infty} a_i x^{i+c-2} = 0$$

Taking the coefficient of the term x^{c-2} : $[c(c-1) + p_0 c + q_0] a_0 = 0$, which

yields the indicial equation $c(c-1) + p_0 c + q_0 = 0$ from which two roots

$c = c_1$ and $c = c_2$ are obtained. The solution $y(x)$ is expressed as

$y(x) = B_1 y_1(x) + B_2 y_2(x)$, where $y_1(x)$ and $y_2(x)$ depend on the following

four cases:

1. Both $c_1 (> c_2)$ and c_2 are real and $c_1 - c_2$ neither zero nor a positive integer.

$$y_1(x) = |x|^{c_1} \left[1 + \sum_{i=1}^{\infty} a_i x^i \right], \quad y_2(x) = |x|^{c_2} \left[1 + \sum_{i=1}^{\infty} b_i x^i \right]$$

2. Both $c_1(>c_2)$ and c_2 are real and $c_1 - c_2$ equal to a positive integer.

$$y_1(x) = |x|^{c_1} [1 + \sum_{i=1}^{\infty} a_i x^i], \quad y_2(x) = A y_1(x) \ln|x| + |x|^{c_2} \sum_{i=1}^{\infty} b_i x^i$$

3. Both c_1 and c_2 are real and $c_1 = c_2$.

$$y_1(x) = |x|^{c_1} [1 + \sum_{i=1}^{\infty} a_i x^i], \quad y_2(x) = y_1(x) \ln|x| + |x|^{c_2} \sum_{i=1}^{\infty} b_i x^i$$

4. $c_1 = \lambda + \sqrt{-1}\mu$ and $c_1 = \lambda - \sqrt{-1}\mu$

$$y_1(x) = \operatorname{Re}\{|x|^{c_1} [1 + \sum_{i=1}^{\infty} a_i x^i]\}, \quad y_2(x) = \operatorname{Im}\{|x|^{c_1} [1 + \sum_{i=1}^{\infty} a_i x^i]\}.$$

Ex. Solve $2xy'' + (x+1)y' + y = 0$

Sol. Let $y(x) = \sum_{i=0}^{\infty} a_i x^{i+c}$,

$$2 \sum_{i=0}^{\infty} (i+c)(i+c-1) a_i x^{i+c-1} + (x+1) \sum_{i=0}^{\infty} (i+c) a_i x^{i+c-1} + \sum_{i=0}^{\infty} a_i x^{i+c} = 0$$

$$\sum_{i=0}^{\infty} 2(i+c)(2i+2c-1) a_i x^{i+c-1} + \sum_{i=0}^{\infty} (i+c+1) a_i x^{i+c} = 0$$

$$c(2c-1) a_0 x^{c-1} + \sum_{i=1}^{\infty} (i+c)(2i+2c-1) a_i x^{i+c-1} + \sum_{i=0}^{\infty} (i+c+1) a_i x^{i+c} = 0$$

Introduce $i = j+1$

$$c(2c-1) a_0 x^{c-1} + \sum_{j=0}^{\infty} (j+1+c)(2j+2c+1) a_{j+1} x^{j+c} + \sum_{i=0}^{\infty} (i+c+1) a_i x^{i+c} = 0$$

$$c(2c-1) a_0 x^{c-1} + \sum_{j=0}^{\infty} (j+1+c)[(2j+2c+1) a_{j+1} + a_j] x^{j+c} = 0$$

$$\rightarrow c=0, \quad c=\frac{1}{2}, \quad a_{j+1} = -\frac{1}{1+2c+2j} a_j$$

$$\text{Case } c=0, \quad a_{j+1} = -\frac{1}{1+2j} a_j, \quad a_1 = -a_0, \quad a_2 = -\frac{1}{3} a_1 = \frac{1}{3} a_0, \quad a_3 = -\frac{1}{5} a_2 = -\frac{1}{15} a_0$$

$$y_1(x) = a_0 (1 - x + \frac{x^2}{3} - \frac{x^3}{15} + \dots)$$

$$\text{Case } c=\frac{1}{2}, \quad a_{j+1} = -\frac{1}{2(1+j)} a_j, \quad a_1 = -\frac{1}{2} a_0, \quad a_2 = -\frac{1}{2^2} a_1 = \frac{1}{2^3} a_0$$

$$y_2(x) = a_0 x^{1/2} (1 - \frac{1}{2} x + \frac{x^2}{8} + \dots)$$

Finally, $y(x) = a_0(1 - x + \frac{x^2}{3} - \frac{x^3}{15} + \dots) + b_0x^{1/2}(1 - \frac{1}{2}x + \frac{x^2}{8} + \dots)$

Ex. $x^2y'' + x(2+x)y' - 2y = 0 \rightarrow y'' + \frac{1}{x}(2+x)y' - \frac{2}{x^2}y = 0$, $p(0) = 2$, $q(0) = -2$

The indicial equation is $c(c-1) + 2c - 2 = 0 \rightarrow c^2 + c - 2 = 0 \rightarrow c_1 = 1$, $c_2 = -2$

$c_1 = 1$, $y_1(x) = \sum_{i=0}^{\infty} a_i x^{i+1}$

$$\sum_{i=1}^{\infty} (i+1)ia_i x^{i+1} + x(2+x) \sum_{i=0}^{\infty} (i+1)a_i x^i - 2 \sum_{i=0}^{\infty} a_i x^{i+1} = 0$$

$$\sum_{i=1}^{\infty} (i+1)ia_i x^{i+1} + 2 \sum_{i=0}^{\infty} ia_i x^{i+1} + \sum_{i=0}^{\infty} (i+1)a_i x^{i+2} = 0$$

$$\sum_{i=1}^{\infty} (i+3)ia_i x^{i+1} + \sum_{i=0}^{\infty} (i+1)a_i x^{i+2} = 0 \rightarrow \sum_{i=0}^{\infty} (i+4)(i+1)a_{i+1} x^{i+2} + \sum_{i=0}^{\infty} (i+1)a_i x^{i+2} = 0$$

$$\rightarrow \sum_{i=0}^{\infty} (i+1)[(i+4)a_{i+1} + a_i] x^{i+2} = 0 \rightarrow a_{i+1} = -\frac{1}{i+4} a_i$$

$$a_1 = -\frac{1}{4}a_0, \quad a_2 = -\frac{1}{5}a_0 = \frac{1}{4 \cdot 5}a_0, \quad a_3 = -\frac{1}{6}a_2 = -\frac{1}{4 \cdot 5 \cdot 6}a_0, \text{ etc.}$$

$$y_1(x) = x(1 - \frac{1}{4}x + \frac{1}{4 \cdot 5}x^2 - \frac{1}{4 \cdot 5 \cdot 6}x^3 + \dots)$$

$c_2 = -2$, $y_2(x) = Ay_1(x) \ln x + x^{-2} \sum_{i=0}^{\infty} b_i x^i$

Then $y_2'(x) = Ay_1'(x) \ln x + A \frac{1}{x} y_1(x) + \sum_{i=0}^{\infty} (i-2)b_i x^{i-3}$

$$y_2''(x) = Ay_1''(x) \ln x + \frac{2}{x} Ay_1'(x) - A \frac{1}{x^2} y_1(x) + \sum_{i=0}^{\infty} (i-2)(i-3)b_i x^{i-4}$$

Substituting $y_2(x)$, $y_2'(x)$ and $y_2''(x)$ into $x^2y'' + x(2+x)y' - 2y = 0$ yields

$$A[x^2y_1'' + x(2+x)y_1' - 2y_1] \ln x + A(y_1 + xy_1 + 2xy_1')$$

$$+ \sum_{i=0}^{\infty} (i-3)(i-2)b_i x^{i-2} + \sum_{i=0}^{\infty} 2(i-2)b_i x^{i-2} + \sum_{i=0}^{\infty} (i-2)b_i x^{i-1} - \sum_{i=0}^{\infty} 2b_i x^{i-2} = 0$$

Or

$$A(y_1 + xy_1 + 2xy_1') + \sum_{i=0}^{\infty} i(i-3)b_i x^{i-2} + \sum_{i=0}^{\infty} (i-2)b_i x^{i-1} = 0$$

$$A(y_1 + xy_1 + 2xy_1') + \sum_{i=1}^{\infty} i(i-3)b_i x^{i-2} + \sum_{i=0}^{\infty} (i-2)b_i x^{i-1} = 0$$

Let $i = j+1$

$$A(y_1 + xy_1 + 2xy_1') + \sum_{j=0}^{\infty} (j+1)(j-2)b_{j+1}x^{j-1} + \sum_{i=0}^{\infty} (i-2)b_i x^{i-1} = 0$$

$$A(y_1 + xy_1 + 2xy_1') + \sum_{i=0}^{\infty} (i-2)[(i+1)b_{i+1} + b_i]x^{i-1} = 0$$

Substituting $y_1(x) = \sum_{i=0}^{\infty} a_i x^{i+1}$ into the above equation yields

$$A[\sum_{i=0}^{\infty} (2i+3)a_i x^{i+1} + \sum_{i=0}^{\infty} a_i x^{i+2}] + \sum_{i=0}^{\infty} (i-2)[(i+1)b_{i+1} + b_i]x^{i-1} = 0 \quad \text{or}$$

$$A[3a_0 x + \sum_{i=1}^{\infty} (2i+3)a_i x^{i+1} + \sum_{i=0}^{\infty} a_i x^{i+2}] + (-2)(b_1 + b_0)x^{-1} + (-1)(2b_2 + b_1) + \sum_{i=3}^{\infty} (i-2)[(i+1)b_{i+1} + b_i]x^{i-1} = 0$$

Or

$$A[3a_0 x + \sum_{j=0}^{\infty} (2j+5)a_{j+1}x^{j+2} + \sum_{i=0}^{\infty} a_i x^{i+2}] + (-2)(b_1 + b_0)x^{-1} + (-1)(2b_2 + b_1) + \sum_{k=0}^{\infty} (k+1)[(k+4)b_{k+4} + b_{k+3}]x^{k+2} = 0$$

which implies

$$A=0, \quad b_1 = -b_0, \quad b_2 = \frac{1}{2}b_0, \quad b_{k+4} = -\frac{1}{k+4}b_{k+3} \quad \text{or} \quad b_4 = -\frac{1}{4}b_3, \quad b_5 = \frac{1}{4 \cdot 5}b_3, \\ b_6 = -\frac{1}{4 \cdot 5 \cdot 6}b_3, \text{ etc.}$$

$$\text{So, } y_2(x) = b_0(x^{-2} - x^{-1} + \frac{1}{2}) + b_3x(1 - \frac{x}{4} + \frac{x^2}{4 \cdot 5} - \frac{x^3}{4 \cdot 5 \cdot 6} + \dots)$$

Since the second term $b_3x(1 - \frac{x}{4} + \frac{x^2}{4 \cdot 5} - \frac{x^3}{4 \cdot 5 \cdot 6} + \dots)$ of $y_2(x)$ is linearly

dependent on $y_1(x)$, then the term can be omitted and $y_2(x)$ can be expressed as

$$y_2(x) = b_0(x^{-2} - x^{-1} + \frac{1}{2})$$

which is linearly independent on $y_1(x)$.

The complete solution of $y(x)$ is $y(x) = c_1 y_1(x) + c_2 y_2(x)$.

$$\text{Ex. } x^2 y'' + (x^2 - x)y' + y = 0 \quad \text{or} \quad y'' + \frac{x^2 - x}{x^2} y' + \frac{1}{x^2} y = 0$$

$x = 0$ is the regular singular point of the problem.

$\lim_{x \rightarrow 0} (x \frac{x^2 - x}{x^2}, x^2 \frac{1}{x^2}) = (-1, 1)$, The indicial equation is $c(c-1) - c + 1 = 0$, $c = 1$,

1. One solution is expressed as $y_1(x) = \sum_{i=0}^{\infty} a_i x^{i+1}$.

$$\sum_{i=0} [i(i+1)+1] a_i x^{i+1} + \sum_{i=0} (i+1) a_i x^{i+2} - \sum_{i=0} (i+1) a_i x^{i+1} = 0$$

$$\sum_{i=0} i^2 a_i x^{i+1} + \sum_{i=0} (i+1) a_i x^{i+2} = 0 \rightarrow \sum_{i=1} i^2 a_i x^{i+1} + \sum_{i=0} (i+1) a_i x^{i+2} = 0 \rightarrow$$

$$\sum_{i=0} (i+1)^2 a_{i+1} x^{i+2} + \sum_{i=0} (i+1) a_i x^{i+2} = 0 \rightarrow \sum_{i=0} (i+1) [(i+1) a_{i+1} + a_i] x^{i+2} = 0$$

$$\rightarrow a_{i+1} = -\frac{1}{i+1} a_i : a_1 = -a_2, a_2 = -\frac{1}{2} a_1 = \frac{1}{2} a_0, a_3 = -\frac{1}{3} a_2 = -\frac{1}{3!} a_0, \text{ etc.}$$

$$y_1(x) = x(1 - x + \frac{x^2}{2} - \frac{x^3}{3!} + \dots) = x e^{-x}$$

$$y_2(x) = y_1(x) \ln x + x \sum_{i=1} b_i x^i \rightarrow y_2'(x) = y_1'(x) \ln x + \frac{1}{x} y_1(x) + \sum_{i=1} (i+1) b_i x^i$$

$$\rightarrow y_2''(x) = y_1''(x) \ln x + \frac{2}{x} y_1'(x) - \frac{1}{x^2} y_1(x) + \sum_{i=1} i(i+1) b_i x^{i-1}$$

$$x^2 y_2''(x) = x^2 y_1''(x) \ln x + 2x y_1'(x) - y_1(x) + \sum_{i=1} i(i+1) b_i x^{i+1}$$

$$x^2 y_2'(x) = x^2 y_1'(x) \ln x + x y_1(x) + \sum_{i=1} (i+1) b_i x^{i+2}$$

$$-x y_2'(x) = -x y_1'(x) \ln x - y_1(x) - \sum_{i=1} (i+1) b_i x^{i+1}$$

$$y_2(x) = y_1(x) \ln x + x \sum_{i=1} b_i x^i$$

$$x^2 y'' + (x^2 - x) y' + y = 0 \rightarrow$$

$$(x^2 y_1'' + x^2 y_1' - x y_1' + y_1) \ln x + 2x y_1' - 2y_1 + x y_1 + \sum_{i=1} i(i+1) b_i x^{i+1} + \sum_{i=1} (i+1) b_i x^{i+2}$$

$$- \sum_{i=1} (i+1) b_i x^{i+1} + x \sum_{i=1} b_i x^i = 0 \rightarrow$$

$$2x y_1' - 2y_1 + x y_1 + \sum_{i=1} i^2 b_i x^{i+1} + \sum_{i=1} (i+1) b_i x^{i+2} = 0$$

$$2x y_1' - 2y_1 + x y_1 + b_1 x^2 + \sum_{i=2} i^2 b_i x^{i+1} + \sum_{i=1} (i+1) b_i x^{i+2} = 0$$

$$2xy'_1 - 2y_1 + xy_1 + b_1x^2 + \sum_{i=1} (i+1)[(i+1)b_{i+1} + b_i]x^{i+2} = 0$$

$$(2x - 4x^2 + 3x^3 - \frac{4}{3}x^4 + \dots) - (2x - 2x^2 + \frac{x^3}{2} - \frac{x^4}{3} + \dots) + (x^2 - x^3 + \frac{x^4}{2} + \dots) + b_1x^2 + \sum_{i=1} (i+1)[(i+1)b_{i+1} + b_i]x^{i+2} = 0$$

$$(-x^2 + x^3 - \frac{x^4}{2} + \dots) + b_1x^2 + \sum_{i=1} (i+1)[(i+1)b_{i+1} + b_i]x^{i+2} = 0$$

$$\rightarrow b_1 = 1, \quad 2(2b_2 + b_1) = -1, \quad 3(3b_3 + b_2) = \frac{1}{2} \rightarrow b_1 = 1, \quad b_2 = -3/4, \quad b_3 = 11/36.$$

$$y_2(x) = y_1(x) \ln x + x \sum_{i=1} b_i x^i$$

$$\text{Ex. } x^2 y'' - xy' + 10y = 0 \quad \text{or} \quad y'' - \frac{1}{x} y' + \frac{10}{x^2} y = 0, \quad p(0) = -1, \quad q(0) = 10$$

The indicial equation $c(c-1) - c + 10 = 0$ yields $c = 1 \pm 3j$, $j = \sqrt{-1}$.

$$y(x) = \sum_{i=0} a_i x^{i+c} \rightarrow \sum_{i=0} [(i+c)(i+c-1) - (i+c) + 10] a_i x^{i+c} = 0 \rightarrow$$

$$\sum_{i=0} [(i+c)(i+c-2) + 10] a_i x^{i+c} = 0 \rightarrow [(i+c)(i+c-2) + 10] a_i = 0$$

$$(i+c)(i+c-2) + 10 \neq 0 \quad \text{for } i = 2, \dots \rightarrow a_i = 0 \quad \text{for } i = 2, \dots$$

$$i = 0, \quad [c(c-2) + 10] a_0 = 0 \rightarrow a_0 \neq 0 \quad (\text{nontrivial solution})$$

$$\text{So, } y(x) = a_0 x^{1+3j} + b_0 x^{1-3j} = a_0 x \exp[\ln x^{3j}] + b_0 x \exp[\ln x^{-3j}]$$

$$= a_0 x \exp[j3 \ln x] + a_0 x \exp[-j3 \ln x]$$

$$= a_0 x [\cos(3 \ln x) + j \sin(3 \ln x)] + b_0 x [\cos(3 \ln x) - j \sin(3 \ln x)]$$

$$= (a_0 + b_0) x \cos(3 \ln x) + j(a_0 - b_0) x \sin(3 \ln x)$$

$$= Ax \cos(3 \ln x) + Bx \sin(3 \ln x)$$

Hw Find two linearly independent solutions and the first four terms of the following equations

$$1. \quad 3x^2 y'' - xy' + (x+1)y = 0$$

$$2. \quad (x-2)^2 y'' - (x-2)y' + (x^2 - 4x + 5)y = 0$$

§ Bessel function of the first kind $J_n(x)$

Bessel's equation $x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - v^2)y = 0 \quad v \geq 0$

Or $\lambda^2 x^2 \frac{d^2 y}{d(\lambda x)^2} + \lambda x \frac{dy}{d(\lambda x)} + (\lambda^2 x^2 - v^2)y = 0$

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (\lambda^2 x^2 - v^2)y = 0$$

$$\frac{d^2 y}{dx^2} + \frac{1}{x} \frac{dy}{dx} + \frac{x^2 - v^2}{x^2} y = 0 \rightarrow p(0) = 1, \quad q(0) = -v^2$$

The indicial equation $c(c-1) + p(0)c + q(0) = 0$ yields $c = \pm v$

$$y(x) = \sum_{i=0} a_i x^{i+v}, \quad \sum_{i=0} [(i+v)(i+v-1)a_i + (i+v)a_i - v^2 a_i] x^{i+v} + \sum_{i=0} a_i x^{i+v+2} = 0$$

$$\sum_{i=0} (i+2v)ia_i x^{i+v} + \sum_{i=0} a_i x^{i+v+2} = 0,$$

$$(1+2v)a_1 x^{1+v} + \sum_{i=2} (i+2v)ia_i x^{i+v} + \sum_{i=0} a_i x^{i+v+2} = 0$$

$$(1+2v)a_1 x^{1+v} + \sum_{i=0} [(i+2+2v)(i+2)a_{i+2} + a_i] x^{i+v+2} = 0$$

$$a_1 = 0, \quad a_{i+2} = -\frac{1}{(i+2+2v)(i+2)} a_i, \quad a_2 = \frac{-1}{2^2(1+v)} a_0,$$

$$a_4 = -\frac{1}{2^3(2+v)} a_2 = \frac{(-1)^2}{2^5(1+v)(2+v)} a_0$$

$$\text{Let } a_0 = \frac{1}{2^v \Gamma(1+v)}, \quad y(x) \triangleq J_v(x) = x^v \sum_{m=0} \frac{(-1)^m x^{2m}}{2^{2m+v} m! \Gamma(m+1+v)}, \quad x \geq 0$$

$J_v(x)$ is the Bessel function of the first kind of order v

$$J_{-v}(x) = x^{-v} \sum_{m=0} \frac{(-1)^m x^{2m}}{2^{2m-v} m! \Gamma(m+1-v)}$$

$$J_n(x) = \sum_{m=0} \frac{(-1)^m x^{2m+n}}{2^{2m+n} m! (m+n)!}$$

The gamma function $\Gamma(x)$ is defined as $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$

with the property $\Gamma(x+1) = x\Gamma(x)$, $\Gamma(n+1) = n!$

When v is not an integer, the general solution of

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - v^2)y = 0$$

is $y(x) = C_1 J_v(x) + C_2 J_{-v}(x)$

and the general solution of $x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (\lambda^2 x^2 - v^2)y = 0$ is

$$y(x) = C_1 J_v(\lambda x) + C_2 J_{-v}(\lambda x)$$

Properties

$$1. J_{-n}(x) = (-1)^n J_n(x)$$

$$\begin{aligned} J_{-n}(x) &= \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m-n}}{2^{2m-n} m!(m-n)!} = \sum_{m=n}^{\infty} \frac{(-1)^m x^{2m-n}}{2^{2m-n} m!(m-n)!} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^{k+n} x^{2(k+n)-n}}{2^{2(k+n)-n} (k+n)!k!} = (-1)^n \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+n}}{2^{2k+n} (k+n)!k!} = (-1)^n J_n(x) \end{aligned}$$

$$2. \frac{d}{dx} J_0(x) = -J_1(x)$$

$$J_0(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m} m!m!},$$

$$\frac{d}{dx} J_0(x) = \sum_{m=0}^{\infty} \frac{(-1)^m 2mx^{2m-1}}{2^{2m} m!m!} = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m-1}}{2^{2m-1} m!(m-1)!} = J_{-1}(x) = -J_1(x)$$

$$3. \frac{d}{dx} (x^v J_v) = x^v J_{v-1}(x) \quad \text{or} \quad \int x^v J_{v-1}(x) dx = x^v J_v + c$$

$$\frac{d}{dx} (x^v J_v) = \sum_{m=0}^{\infty} \frac{(-1)^m 2(m+v)x^{2m+2v-1}}{2^{2m+v} m!\Gamma(m+1+v)} = x^v \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+v-1}}{2^{2m+v-1} m!\Gamma(m+v)} = x^v J_{v-1}$$

$$4. \frac{d}{dx} (x^{-v} J_v) = -x^{-v} J_{v-1}(x) \quad \text{or} \quad \int x^{-v} J_{v-1}(x) dx = -x^{-v} J_v + c$$

$$5. J_{v-1}(x) + J_{v+1}(x) = \frac{2v}{x} J_v(x)$$

$$6. J_{v-1}(x) - J_{v+1}(x) = 2J'_v(x)$$

Ex. Express $J_4(x)$ in terms of $J_0(x)$ and $J_1(x)$

$$\begin{aligned} \text{Ans. } J_4 &= J_{3+1} = \frac{6}{x} J_3 - J_2 = \frac{6}{x} \left(\frac{4}{x} J_2 - J_1 \right) - J_2 = \left(\frac{24}{x^2} - 1 \right) J_2 - \frac{6}{x} J_1 \\ &= \left(\frac{24}{x^2} - 1 \right) \left(\frac{2}{x} J_1 - J_0 \right) - \frac{6}{x} J_1 = \left(\frac{48}{x^3} - \frac{8}{x} \right) J_1 + \left(1 - \frac{24}{x^2} \right) J_0 \end{aligned}$$

Ex. Evaluate $\int (x^2 + \frac{1}{x}) J_1 dx$

Ans. $\int x^\nu J_{\nu-1}(x) dx = x^\nu J_\nu + c$, $\int x^{-\nu} J_{\nu-1}(x) dx = -x^{-\nu} J_\nu + c$

$$\begin{aligned} \int (x^2 + \frac{1}{x}) J_1 dx &= \int x^2 J_1 dx + \int \frac{1}{x} J_1 dx = x^2 J_2 + \int x^{-2} (x J_1) dx + c \\ &= x^2 J_2 - \int x J_1 dx^{-1} + c = x^2 J_2 - x^{-1} (x J_1) + \int x^{-1} \frac{d}{dx} (x J_1) dx + c \\ &= x^2 J_2 - J_1 + \int x^{-1} x J_0 dx + c = x^2 J_2 - J_1 + \int J_0 dx + c \end{aligned}$$

Ex. Evaluate $\int x^3 J_0 dx$

$$\begin{aligned} \text{Ans. } \int x^3 J_0 dx &= \int x^2 (x J_0) dx = \int x^2 \frac{d}{dx} (x J_1) dx = x^2 x J_1 - \int 2x x J_1 dx \\ &= x^3 J_1 - 2 \int x^2 J_1 dx = x^3 - 2x^2 J_2 + c \end{aligned}$$

§ Bessel Function $J_{\pm n/2}(x)$

By substituting $u = x^{1/2} y$ into the Bessel equation

$$x^2 y'' + xy' + (x^2 - v^2)y = 0$$

$$\text{yields } u'' + (1 - \frac{4v^2 - 1}{4x^2})u = 0.$$

$$y = x^{-1/2} u, \quad y' = (x^{-1/2} u)' = -\frac{1}{2} x^{-3/2} u + x^{-1/2} u', \quad y'' = \frac{3}{4} x^{-5/2} u - x^{-3/2} u' + x^{-1/2} u''$$

$$x^2 y'' = \frac{3}{4} x^{-1/2} u - x^{1/2} u' + x^{3/2} u'', \quad xy' = -\frac{1}{2} x^{-1/2} u + x^{1/2} u',$$

$$(x^2 - v^2)y = (x^2 - v^2)x^{-1/2}u = x^{3/2}u - v^2 x^{-1/2}u$$

$$x^{3/2}u'' + [x^{3/2} + (\frac{1}{4} - v^2)x^{-1/2}]u = 0 \rightarrow u'' + [1 - \frac{4v^2 - 1}{4x^2}]u = 0$$

$$\text{Let } v = \frac{1}{2}, \quad u'' + u = 0, \quad u = a_1 \cos x + a_2 \sin x, \quad y = x^{-1/2}(a_1 \cos x + a_2 \sin x)$$

$$\text{Take } a_1 = a_2 = \sqrt{2/\pi} \rightarrow J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x, \quad J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$$

$$\text{From } J_{\nu+1} + J_{\nu-1} = \frac{2\nu}{x} J_\nu, \text{ we obtain}$$

$$J_{3/2} = \frac{1}{x} J_{1/2} - J_{-1/2} = \sqrt{\frac{2}{\pi x}} \left(\frac{\sin x}{x} - \cos x \right),$$

$$J_{-3/2} = -\frac{1}{x} J_{-1/2} - J_{1/2} = -\sqrt{\frac{2}{\pi x}} \left(\sin x + \frac{\cos x}{x} \right)$$

Hw. 1. Prove the following equations

a. $\frac{d}{dx}[x^\nu J_\nu(\lambda x)] = \lambda x^\nu J_{\nu-1}(\lambda x)$, b. $\frac{d}{dx}[x^{-\nu} J_\nu(\lambda x)] = -\lambda x^{-\nu} J_{\nu+1}(\lambda x)$,

c. $\frac{d}{dx}[J_\nu(\lambda x)] = \lambda J_{\nu-1}(\lambda x) - \frac{\nu}{x} J_\nu(\lambda x)$, d. $\frac{d}{dx}[J_\nu(\lambda x)] = -\lambda J_{\nu+1}(\lambda x) + \frac{\nu}{x} J_\nu(\lambda x)$

e. $\frac{d}{dx}[J_\nu(\lambda x)] = \frac{\lambda}{2} [J_{\nu-1}(\lambda x) - J_{\nu+1}(\lambda x)]$, f. $J_\nu(\lambda x) = \frac{\lambda x}{2\nu} [J_{\nu-1}(\lambda x) + J_{\nu+1}(\lambda x)]$

2. Express $\int J_5(x) dx$ in terms of $J_0(x)$, $J_2(x)$ and $J_4(x)$

3. Express $\int x^2 J_0(x) dx$ in terms of $\int J_0(x) dx$.

§ Bessel functions of the second order kind $Y_\nu(x)$

$$x^2 y'' + xy' + (x^2 - \nu^2)y = 0$$

$$x^2 J_\nu'' + x J_\nu' + (x^2 - \nu^2) J_\nu = 0 \quad \text{and} \quad x^2 J_{-\nu}'' + x J_{-\nu}' + (x^2 - \nu^2) J_{-\nu} = 0$$

$$\frac{\partial}{\partial \nu} [x^2 J_\nu'' + x J_\nu' + (x^2 - \nu^2) J_\nu] = 0 \rightarrow x^2 \frac{\partial J_\nu''}{\partial \nu} + x \frac{\partial J_\nu'}{\partial \nu} + (x^2 - \nu^2) \frac{\partial J_\nu}{\partial \nu} = 2\nu J_\nu \quad (1)$$

$$\frac{\partial}{\partial \nu} [x^2 J_{-\nu}'' + x J_{-\nu}' + (x^2 - \nu^2) J_{-\nu}] = 0 \rightarrow x^2 \frac{\partial J_{-\nu}''}{\partial \nu} + x \frac{\partial J_{-\nu}'}{\partial \nu} + (x^2 - \nu^2) \frac{\partial J_{-\nu}}{\partial \nu} = 2\nu J_{-\nu} \quad (2)$$

$$\text{Eq. (1)} - (-1)^n \times \text{Eq. (2)} \rightarrow$$

$$\left[x^2 \frac{d^2}{dx^2} + x \frac{d}{dx} + (x^2 - \nu^2) \right] \frac{\partial}{\partial \nu} [J_\nu - (-1)^n J_{-\nu}] = 2\nu [J_\nu - (-1)^n J_{-\nu}]$$

$$\lim_{\nu \rightarrow n} [J_\nu - (-1)^n J_{-\nu}] = 0, \text{ So, } \lim_{\nu \rightarrow n} \frac{\partial}{\partial \nu} [J_\nu - (-1)^n J_{-\nu}] \text{ also is a solution of the}$$

Bessel equation.

Define the second kind of Bessel function $Y_\nu(x)$ as

$$Y_\nu(x) = \frac{\cos \nu \pi J_\nu(x) - J_{-\nu}(x)}{\sin \nu \pi}$$

Case 1 $\nu \neq n$, $Y_\nu(x)$ is the linear combination of $J_\nu(x)$ and $J_{-\nu}(x)$.

Case 2 $\nu = n$

$$\begin{aligned}
Y_n(x) &= \lim_{v \rightarrow n} \frac{\cos v\pi J_v(x) - J_{-v}(x)}{\sin v\pi} = \lim_{v \rightarrow n} \frac{1}{\pi \cos v\pi} [-\pi \sin v\pi J_v + \cos v\pi \frac{\partial J_v}{\partial v} - \frac{\partial J_{-v}}{\partial v}] \\
&= \frac{1}{\pi} \lim_{v \rightarrow n} [\frac{\partial J_v}{\partial v} - \frac{1}{\cos v\pi} \frac{\partial J_{-v}}{\partial v}] = \frac{1}{\pi} \lim_{v \rightarrow n} [\frac{\partial J_v}{\partial v} - (-1)^n \frac{\partial J_{-v}}{\partial v}] \\
\pi Y_n(x) &= 2J_n(x)(\gamma + \ln \frac{x}{2}) + \frac{x^n}{\pi} \sum_{i=0}^{\infty} \frac{(-1)^{m-1}(h_i + h_{i+n})}{2^{2i+n} i!(i+n)!} x^{2m} \\
&\quad - \frac{1}{\pi x^n} \sum_{i=0}^{n-1} \frac{(n-m-1)!}{2^{2m-n} m!} x^{2m}
\end{aligned}$$

Where

$$h_i = 1 + \frac{1}{2} + \dots + \frac{1}{i}, \quad \gamma = \lim_{i \rightarrow \infty} (1 + \frac{1}{2} + \dots + \frac{1}{i} - \ln i)$$

$$Y_n(x) = (-1)^n Y_n(x)$$

The general solution of Bessel equation is expressed as

$$y(x) = AJ_v(x) + BY_v(x).$$

§ Modified Bessel functions $I_v(x)$ and $K_v(x)$

$$x^2 y'' + xy' - (x^2 + v^2)y = 0$$

$$J_v(jx) = \sum_{i=0}^{\infty} \frac{(-1)^i (jx)^{2i+v}}{2^{2i+v} i! \Gamma(i+1+v)} = j^v \sum_{i=0}^{\infty} \frac{x^{2i+v}}{2^{2i+v} i! \Gamma(i+1+v)}, \quad j = \sqrt{-1}$$

Modified Bessel function of the first kind

$$I_v(x) = \sum_{i=0}^{\infty} \frac{x^{2i+v}}{2^{2i+v} i! \Gamma(i+1+v)}$$

$$J_v(jx) = j^v I_v(x) \rightarrow I_v(x) = j^{-v} J_v(jx), \quad I_{-v}(x) = j^v J_{-v}(jx)$$

If v is an integer, then the modified Bessel function of the second kind is

$$K_v(x) = \frac{\pi}{2} \frac{I_{-v}(x) - I_v(x)}{\sin(v\pi)} \quad \text{and} \quad K_n(x) = \frac{\pi}{2} \lim_{v \rightarrow n} \frac{I_{-v}(x) - I_v(x)}{\sin(v\pi)}$$

The general solution of $x^2 y'' + xy' - (x^2 + v^2)y = 0$ is $y(x) = AI_v(x) + BK_v(x)$.

Furthermore, the general solution of $x^2 y'' + xy' - (\lambda^2 x^2 + v^2)y = 0$ is

$$y(x) = AI_v(\lambda x) + BK_v(\lambda x).$$

Hw

1. Use the definition of $I_\nu(x)$ to show that

a. $I_{\nu-1}(x) - I_{\nu+1}(x) = \frac{2\nu}{x} I_\nu(x), \quad \nu \geq 1$

b. $I_{\nu-1}(x) + I_{\nu+1}(x) = 2I'_\nu(x), \quad \nu \geq 1$

§ Orthogonality of Bessel Functions

$$xJ''_\nu(\alpha_n x) + J'_\nu(\alpha_n x) + (\alpha_n^2 x - \frac{\nu^2}{x})J_\nu(\alpha_n x) = 0 \quad (1)$$

$$xJ''_\nu(\alpha_m x) + J'_\nu(\alpha_m x) + (\alpha_m^2 x - \frac{\nu^2}{x})J_\nu(\alpha_m x) = 0 \quad (2)$$

With one of the following three kinds of boundary conditions

1. $J_\nu(\alpha_i a) = 0, \quad i = 1, \dots$

2. $J'_\nu(\alpha_i a) = 0, \quad i = 1, \dots$

3. $kJ_\nu(\alpha_i a) + J'_\nu(\alpha_i a) = 0$

$$\begin{aligned} \int_0^a J_\nu(\alpha_m x) \times (1) dx &\rightarrow \int_0^a J_\nu(\alpha_m x) [xJ''_\nu(\alpha_n x) + J'_\nu(\alpha_n x) + (\alpha_n^2 x - \frac{\nu^2}{x})J_\nu(\alpha_n x)] dx = 0 \\ &\rightarrow xJ'_\nu(\alpha_n x) J_\nu(\alpha_m x) \Big|_0^a - \int_0^a J'_\nu(\alpha_n x) d[xJ_\nu(\alpha_m x)] + \int_0^a J_\nu(\alpha_m x) J'_\nu(\alpha_n x) dx \\ &\quad + \alpha_n^2 \int_0^a xJ_\nu(\alpha_m x) J_\nu(\alpha_n x) dx - \nu^2 \int_0^a \frac{1}{x} J_\nu(\alpha_m x) J_\nu(\alpha_n x) dx = 0 \\ &\rightarrow - \int_0^a J'_\nu(\alpha_n x) J_\nu(\alpha_m x) dx - \int_0^a xJ'_\nu(\alpha_n x) dJ_\nu(\alpha_m x) + \int_0^a J_\nu(\alpha_m x) J'_\nu(\alpha_n x) dx \\ &\quad + \alpha_n^2 \int_0^a xJ_\nu(\alpha_m x) J_\nu(\alpha_n x) dx - \nu^2 \int_0^a \frac{1}{x} J_\nu(\alpha_m x) J_\nu(\alpha_n x) dx = 0 \rightarrow \\ &\quad \alpha_n^2 \int_0^a xJ_\nu(\alpha_m x) J_\nu(\alpha_n x) dx = \nu^2 \int_0^a \frac{1}{x} J_\nu(\alpha_m x) J_\nu(\alpha_n x) dx + \int_0^a xJ'_\nu(\alpha_n x) J'_\nu(\alpha_m x) dx \quad (3) \end{aligned}$$

Similarly,

$$\alpha_m^2 \int_0^a xJ_\nu(\alpha_m x) J_\nu(\alpha_n x) dx = \nu^2 \int_0^a \frac{1}{x} J_\nu(\alpha_m x) J_\nu(\alpha_n x) dx + \int_0^a xJ'_\nu(\alpha_n x) J'_\nu(\alpha_m x) dx \quad (4)$$

The above two equations imply

$$(\alpha_m^2 - \alpha_n^2) \int_0^a xJ_\nu(\alpha_m x) J_\nu(\alpha_n x) dx = 0$$

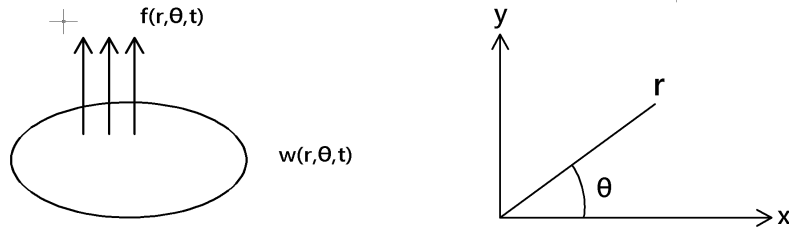
which shows $(\alpha_m^2 - \alpha_n^2) \int_0^a xJ_\nu(\alpha_m x) J_\nu(\alpha_n x) dx = 0$ for $m \neq n$.

Furthermore,

$$\alpha_n^2 \int_0^a x J_v(\alpha_n x) J_v(\alpha_n x) dx = v^2 \int_0^a \frac{1}{x} J_v(\alpha_n x) J_v(\alpha_n x) dx + \int_0^a x J_v'(\alpha_n x) J_v'(\alpha_n x) dx$$

§2 Membrane

2-2 Polar coordinates



Ex. Circular membrane fixed around the boundary $r = a$

Case 1 Free Vibration

$$w(r, \theta, t) = \sin(\omega t) R(r) \Theta(\theta)$$

$$T_0 \left[\Theta \left(\frac{d^2 R}{dr^2} + \frac{1}{r} \frac{dR}{dr} \right) + R \frac{1}{r^2} \frac{d^2 \Theta}{d\theta^2} \right] + \rho \omega^2 R \Theta = 0$$

$$r^2 \left[\frac{1}{R} \left(\frac{d^2 R}{dr^2} + \frac{1}{r} \frac{dR}{dr} \right) + \lambda^2 \right] + \frac{1}{\Theta} \frac{d^2 \Theta}{d\theta^2} = 0, \quad \lambda^2 = \frac{\rho \omega^2}{T_0}$$

$$r^2 \left[\frac{1}{R} \left(\frac{d^2 R}{dr^2} + \frac{1}{r} \frac{dR}{dr} \right) + \lambda^2 \right] = - \frac{1}{\Theta} \frac{d^2 \Theta}{d\theta^2} = k^2$$

$$\Theta = c_1 e^{jk\theta} + c_2 e^{-jk\theta}$$

Due to Θ is a single-value function of θ , i.e.,

$$\Theta = c_1 e^{jk\theta} + c_2 e^{-jk\theta} = c_1 e^{jk(\theta+2m\pi)} + c_2 e^{-jk(\theta+2m\pi)}$$

$$= c_1 e^{jk\theta} e^{j2km\pi} + c_2 e^{-jk\theta} e^{-j2km\pi}$$

which implies k is a positive integer.

$$\Theta = c_1 \cos(m\theta) + c_2 \sin(m\theta)$$

$$r^2 \left[\frac{1}{R} \left(\frac{d^2 R}{dr^2} + \frac{1}{r} \frac{dR}{dr} \right) + \lambda^2 \right] - m^2 = 0 \quad \text{or}$$

$$r^2 \frac{d^2 R}{dr^2} + r \frac{dR}{dr} + (\lambda^2 r^2 - m^2)R = 0 \dots \text{Bessel's equation}$$

$$R(r) = c_3 J_m(\lambda r) + c_4 Y_m(\lambda r)$$

However, $\lim_{r \rightarrow 0} J_m(\lambda r) = \text{finite}$, $\lim_{r \rightarrow 0} Y_m(\lambda r) = -\infty$, $R(r) = c_3 J_m(\lambda r)$.

The roots of $J_m(\lambda a) = 0$ are denoted as $\lambda_{mn} a = j_{mn}$

$$\lambda_{mn}^2 = (j_{mn} / a)^2 = \rho \omega_{mn}^2 / T_0, \quad \omega_{mn}^2 = \lambda_{mn}^2 T_0 / \rho$$

Orthogonality

$$\begin{aligned} r^2 \frac{d^2}{dr^2} J_m\left(\frac{j_{mn}r}{a}\right) + r \frac{d}{dr} J_m\left(\frac{j_{mn}r}{a}\right) + (\lambda_{mn}^2 r^2 - m^2) J_m\left(\frac{j_{mn}r}{a}\right) &= 0 \quad \text{or} \\ r \left[r \frac{d^2}{dr^2} J_m\left(\frac{j_{mn}r}{a}\right) + \frac{d}{dr} J_m\left(\frac{j_{mn}r}{a}\right) \right] + (\lambda_{mn}^2 r^2 - m^2) J_m\left(\frac{j_{mn}r}{a}\right) &= 0 \\ \frac{1}{r} \left\{ \frac{d}{dr} \left[r \frac{d}{dr} J_m\left(\frac{j_{mn}r}{a}\right) \right] \right\} + \left(\lambda_{mn}^2 - \frac{m^2}{r^2} \right) J_m\left(\frac{j_{mn}r}{a}\right) &= 0 \\ \int_0^a J_m\left(\frac{j_{mp}r}{a}\right) \frac{1}{r} \left\{ \frac{d}{dr} \left[r \frac{d}{dr} J_m\left(\frac{j_{mn}r}{a}\right) \right] \right\} r dr + \int_0^a J_m\left(\frac{j_{mp}r}{a}\right) \left(\lambda_{mn}^2 - \frac{m^2}{r^2} \right) J_m\left(\frac{j_{mn}r}{a}\right) r dr &= 0 \\ \int_0^a J_m\left(\frac{j_{mp}r}{a}\right) \left\{ \frac{d}{dr} \left[r \frac{d}{dr} J_m\left(\frac{j_{mn}r}{a}\right) \right] \right\} dr + \int_0^a J_m\left(\frac{j_{mp}r}{a}\right) \left(\lambda_{mn}^2 - \frac{m^2}{r^2} \right) J_m\left(\frac{j_{mn}r}{a}\right) r dr &= 0 \\ \int_0^a J_m\left(\frac{j_{mp}r}{a}\right) \left\{ \frac{d}{dr} \left[r \frac{d}{dr} J_m\left(\frac{j_{mn}r}{a}\right) \right] \right\} dr + \int_0^a J_m\left(\frac{j_{mp}r}{a}\right) \left(\lambda_{mn}^2 - \frac{m^2}{r^2} \right) J_m\left(\frac{j_{mn}r}{a}\right) r dr &= 0 \\ r J_m\left(\frac{j_{mp}r}{a}\right) \frac{d}{dr} J_m\left(\frac{j_{mn}r}{a}\right) \Big|_0^a - \int_0^a r \frac{d}{dr} J_m\left(\frac{j_{mp}r}{a}\right) \frac{d}{dr} J_m\left(\frac{j_{mn}r}{a}\right) dr & \\ + \int_0^a \left(\lambda_{mn}^2 - \frac{m^2}{r^2} \right) J_m\left(\frac{j_{mp}r}{a}\right) J_m\left(\frac{j_{mn}r}{a}\right) r dr &= 0, \\ - \int_0^a r \frac{d}{dr} J_m\left(\frac{j_{mp}r}{a}\right) \frac{d}{dr} J_m\left(\frac{j_{mn}r}{a}\right) dr + \int_0^a \left(\lambda_{mn}^2 - \frac{m^2}{r^2} \right) J_m\left(\frac{j_{mp}r}{a}\right) J_m\left(\frac{j_{mn}r}{a}\right) r dr &= 0 \end{aligned}$$

Similarly,

$$\begin{aligned} - \int_0^a r \frac{d}{dr} J_m\left(\frac{j_{mn}r}{a}\right) \frac{d}{dr} J_m\left(\frac{j_{mp}r}{a}\right) dr + \int_0^a \left(\lambda_{mp}^2 - \frac{m^2}{r^2} \right) J_m\left(\frac{j_{mn}r}{a}\right) J_m\left(\frac{j_{mp}r}{a}\right) r dr &= 0 \\ (\lambda_{mp}^2 - \lambda_{mn}^2) \int_0^a r J_m\left(\frac{j_{mn}r}{a}\right) J_m\left(\frac{j_{mp}r}{a}\right) dr &= 0 \quad \text{implies} \\ \int_0^a r J_m\left(\frac{j_{mn}r}{a}\right) J_m\left(\frac{j_{mp}r}{a}\right) dr &= 0, \quad n \neq p. \end{aligned}$$

Case 2 Forced Vibration

$$\begin{aligned}
& T_0 \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) w + f(r, \theta, t) = \rho \frac{\partial^2 w}{\partial t^2} \\
& w(r, \theta, t) = \sum_{m,n} [A_{mn}(t) \cos(m\theta) + B_{mn}(t) \sin(m\theta)] J_m(j_{mn} r / a) \\
& - \frac{T_0}{\rho} \sum_{m,n} [A_{mn} \cos(m\theta), B_{mn} \sin(m\theta)] \left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{m^2}{r^2} \right) J_m \left(\frac{j_{mn} r}{a} \right) \\
& + \sum_{m,n} \frac{d^2}{dt^2} [A_{mn} \cos(m\theta), B_{mn} \sin(m\theta)] J_m \left(\frac{j_{mn} r}{a} \right) = \frac{1}{\rho} f(r, \theta, t), \\
& - \frac{T_0}{\rho} \sum_n (A_{mn}, B_{mn}) \left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{m^2}{r^2} \right) J_m \left(\frac{j_{mn} r}{a} \right) \\
& + \sum_n \frac{d^2}{dt^2} (A_{mn}, B_{mn}) J_m \left(\frac{j_{mn} r}{a} \right) = \frac{1}{\rho} \frac{1}{\pi} \int_0^{2\pi} f(r, \theta, t) [\cos(m\theta), \sin(m\theta)] d\theta \\
& = (F_m^c, F_s^c)(r, t),
\end{aligned}$$

However,

$$\begin{aligned}
& r^2 \frac{d^2}{dr^2} J_m \left(\frac{j_{mn} r}{a} \right) + r \frac{d}{dr} J_m \left(\frac{j_{mn} r}{a} \right) + (\lambda_{mn}^2 r^2 - m^2) J_m \left(\frac{j_{mn} r}{a} \right) = 0 \\
& \text{Or } \left[\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} + (\lambda_{mn}^2 - \frac{m^2}{r^2}) \right] J_m \left(\frac{j_{mn} r}{a} \right) = 0 \text{ implies} \\
& \left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{m^2}{r^2} \right) J_m \left(\frac{j_{mn} r}{a} \right) = -\lambda_{mn}^2 J_m \left(\frac{j_{mn} r}{a} \right), \\
& \lambda_{mn}^2 \frac{T_0}{\rho} \sum_n (A_{mn}, B_{mn}) J_m \left(\frac{j_{mn} r}{a} \right) + \sum_n \frac{d^2}{dt^2} (A_{mn}, B_{mn}) J_m \left(\frac{j_{mn} r}{a} \right) = (F_m^c, F_s^c)(r, t), \\
& \sum_n \left(\frac{d^2}{dt^2} + \omega_{mn}^2 \right) (A_{mn}, B_{mn}) J_m \left(\frac{j_{mn} r}{a} \right) = (F_m^c, F_s^c)(r, t) \\
& \left(\frac{d^2}{dt^2} + \omega_{mn}^2 \right) (A_{mn}, B_{mn}) = \int_0^a r (F_m^c, F_s^c)(r, t) J_m(j_{mn} r / a) / \int_0^a r J_m^2(j_{mn} r / a) dr \\
& = (F_{mn}^c, F_{mn}^s)(t)
\end{aligned}$$

The initial conditions are

$$\begin{aligned}
& w(r, \theta, 0) = \sum_{m,n} [A_{mn}(0) \cos(m\theta) + B_{mn}(0) \sin(m\theta)] J_m(j_{mn} r / a) \\
& \frac{\partial}{\partial t} w(r, \theta, 0) = \sum_{m,n} [A_{mn}(0) \cos(m\theta) + B_{mn}(0) \sin(m\theta)] J_m(j_{mn} r / a) \\
& (A_{mn}, B_{mn})(0) = \frac{1}{\pi} \int_0^a \int_0^{2\pi} r w(r, \theta, 0) [\cos(m\theta), \sin(m\theta)] J_m(j_{mn} r / a) dr / \\
& \int_0^a r J_m^2(j_{mn} r / a) dr
\end{aligned}$$

$$\frac{d}{dt}(A_{mn}, B_{mn})(0) = \frac{1}{\pi} \int_0^a \int_0^{2\pi} r \frac{\partial}{\partial t} w(r, \theta, 0) [\cos(m\theta), \sin(m\theta)] J_m(j_{mn}r/a) dr /$$

$$\int_0^a r J_m^2(j_{mn}r/a) dr$$

§3 Heat conduction

$$\frac{\partial}{\partial t} \Delta Q = -k \frac{\partial T}{\partial n} \Delta S$$

ΔQ : internal energy, k : coefficient of heat conduction, T : temperature,

n : normal direction, ΔS : surface area.

$$\frac{\partial}{\partial t} \Delta Q = -\rho c \frac{\partial T}{\partial t} \Delta V + f(r, \theta, z, t) \Delta V$$

ΔV : volume, ρ : density, c : specific heat, f : heat source.

$$\frac{\partial}{\partial t} Q = -k \iint \frac{\partial T}{\partial n} dS = -k \iint \nabla T \cdot \bar{n} dS = -k \iiint \nabla \cdot \nabla T \bar{n} dV = -k \iiint \nabla^2 T dV$$

$$= \iiint (f - \rho c \frac{\partial T}{\partial t}) dV$$

$$\rho c \frac{\partial T}{\partial t} - k \nabla^2 T = f(r, \theta, z, t) \quad \text{or} \quad \alpha \frac{\partial T}{\partial t} - \nabla^2 T = f(r, \theta, z, t) / k, \quad \alpha = \rho c / k.$$

Ex. $\frac{\partial u}{\partial t} = k^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} \right)$

with $u(1, \theta, t) = 0$, $u(r, \theta, 0) = (1-r) \cos \theta$, $\frac{\partial}{\partial \theta} u(r, 0, t) = 0$, $\frac{\partial}{\partial \theta} u(r, \pi, t) = 0$

Let $u(r, \theta, t) = T(t)R(r) \cos \theta$

$$R \cos \theta \frac{1}{k^2} \frac{dT}{dt} = T \cos \theta \left(\frac{d^2 R}{dr^2} + \frac{1}{r} \frac{dR}{dr} - \frac{R}{r^2} \right),$$

$$\frac{1}{k^2 T} \frac{dT}{dt} = \frac{1}{R} \left(\frac{d^2 R}{dr^2} + \frac{1}{r} \frac{dR}{dr} - \frac{R}{r^2} \right) \equiv -\alpha^2$$

$$T(t) = e^{-(\alpha k)^2 t}, \quad r^2 \frac{d^2 R}{dr^2} + r \frac{dR}{dr} + (\alpha^2 r^2 - 1)R = 0 \quad \dots \text{Bessel's eq.}$$

$$R(r) = bJ_1(\alpha r) + cY_1(\alpha r), \quad \lim_{r \rightarrow 0} Y_1(\alpha r) \rightarrow \infty, \quad R(r) = bJ_1(\alpha r)$$

The condition $u(1, \theta, t) = 0$ implies $J_1(\alpha) = 0$ whose roots are denoted as

$$\alpha_{1,n}$$

$$u(r, \theta, t) = \cos \theta \sum_{n=1} b_n \exp[-(\alpha_{1,n} k)^2 t] J_1(\alpha_{1,n} r),$$

$$(1-r) \cos \theta = \cos \theta \sum_{n=1} b_n J_1(\alpha_{1,n} r), \quad (1-r) = \sum_{n=1} b_n J_1(\alpha_{1,n} r)$$

$$b_n = \int_0^1 (1-r) r J_1(\alpha_{1,n} r) dr / \int_0^1 r J_1^2(\alpha_{1,n} r) dr$$